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POWER SHAPING CONTROL OF NONLINEAR SYSTEMS: A BENCHMARK EXAMPLE

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Abstract: It is well known that energy balancing control is stymied by the presence of pervasive dissipation. To overcome this problem in electrical circuits, the authors recently proposed the alternative paradigm of *power shaping*—where, as suggested by its name, stabilization is achieved shaping a function akin to power instead of the energy function. In this paper we extend this technique to general nonlinear systems and apply it for the stabilization of the benchmark tunnel diode circuit. It is shown that, in contrast with other techniques recently reported in the literature, e.g. piece-wise approximation of nonlinearities, power shaping yields a simple linear static state feedback that ensures (robust) global asymptotic stability of the desired equilibrium. *Copyright ©2006 IFAC*

Keywords: Passivity-based control, nonlinear control, stability, nonlinear systems.

1. INTRODUCTION AND BACKGROUND

Passive systems constitute a very important class of dynamical systems for which the stored energy cannot exceed the energy supplied to them by the external environment—the difference being the dissipated energy. In view of this energy-balancing feature, it is clear that passivity is intimately related with the property of stability, a *sine qua non* condition for any controller design. Furthermore, invoking the universal principle of energy conservation, it may be argued that all physical systems are passive with respect to some suitably defined port variables that couple the system with the environment. It is not surprising then that, since

the introduction of the first passivity-based controller (PBC) more than two decades ago Takegaki and Arimoto (1981); Ortega and Spong (1989), we have witnessed an ever increasing popularity of passivity as a building block for controller design for all classes of physical systems.

PBC can be used to stabilize a given equilibrium point. In this case we must modify the energy function—that will qualify as a Lyapunov function—to assign a minimum at this point, a step called energy shaping; which, combined with damping injection, constitute the two main stages of PBC (Ortega *et al.*, 1998; van der Schaft, 2000). There are several ways to achieve energy shaping, the most physically appealing being the so-called *energy-balancing* PBC (or control by interconnection) method (Ortega *et al.*, 2002; van der Schaft, 2000). With this procedure the energy function assigned

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to the closed-loop system is the difference between the total energy of the system and the energy supplied by the controller, hence the name energy balancing. Unfortunately, energy balancing PBC is stymied by the presence of pervasive dissipation, that is, the existence of dissipative elements, e.g. resistors, whose power does not vanish at the desired equilibrium point.

To put our contribution in perspective let us briefly recall the principles of energy-balancing control (Ortega *et al.*, 2001). Consider a system whose state space representation is given by¹

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \\ \mathbf{y} &= \mathbf{h}(\mathbf{x})\end{aligned}\quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$ are the input and output vectors, respectively. We assume that the system (1) satisfies the energy-balance inequality, that is, along all trajectories compatible with $\mathbf{u} : [0, t] \rightarrow \mathbb{R}^m$,

$$H(\mathbf{x}(t)) - H(\mathbf{x}(0)) \leq \int_0^t \mathbf{u}^\top(\tau) \mathbf{y}(\tau) d\tau \quad (2)$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is the stored energy function.² In energy-balancing control, we look for a control such that the energy supplied by the controller can be expressed as a function of the state. Indeed, from (2) we see that for any function $\hat{\mathbf{u}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$-\int_0^t \hat{\mathbf{u}}^\top(\mathbf{x}(\tau)) \mathbf{h}(\mathbf{x}(\tau)) d\tau = H_a(\mathbf{x}(t)) - H_a(\mathbf{x}(0)) \quad (3)$$

for some function $H_a : \mathbb{R}^n \rightarrow \mathbb{R}$, the control $\mathbf{u} = \hat{\mathbf{u}}(\mathbf{x}) + \mathbf{v}$ will ensure that the closed-loop system satisfies

$$H_d(\mathbf{x}(t)) - H_d(\mathbf{x}(0)) \leq \int_0^t \mathbf{v}^\top(\tau) \mathbf{y}(\tau) d\tau$$

where $H_d(\mathbf{x}) = H(\mathbf{x}) + H_a(\mathbf{x})$ is the new total energy function. If, furthermore, $\mathbf{x}^* = \arg \min H_d(\mathbf{x})$ then \mathbf{x}^* will be a stable equilibrium of the closed-loop system (with Lyapunov function the difference between the stored and the supplied energies $H_d(\mathbf{x})$).

Unfortunately, as shown in (Ortega *et al.*, 2001), energy-balancing stabilization is stymied by the existence of pervasive dissipation—term which refers to the existence of dissipative elements whose power does not vanish at the desired equilibrium point. More precisely, since solving (3) is equivalent to the solution of the PDE

$$(\dot{H}_a) = \nabla H_a^\top [\mathbf{f} + \mathbf{g}\hat{\mathbf{u}}^\top] = -\hat{\mathbf{u}}^\top \mathbf{h}, \quad (4)$$

and the left hand side is equal to zero at \mathbf{x}^* , it is clear that the method is applicable only to systems verifying $\hat{\mathbf{u}}^\top(\mathbf{x}^*) \mathbf{h}(\mathbf{x}^*) = 0$.

¹ All vectors defined in the paper are *column* vectors, even the gradient of a scalar function that we denote with the operator $\nabla = \partial/\partial \mathbf{x}$. Also, we use $(\cdot)'$ to denote differentiation for functions of scalar arguments. To simplify the expressions, the arguments of all functions will be omitted, and will be explicitly written only the first time that the function is defined.

² Notice that no assumption of non-negativity on $H(\mathbf{x})$ is imposed. If $H(\mathbf{x}) \geq 0$, the system (1) is said to be *passive* with conjugated port variables (\mathbf{u}, \mathbf{y}) .

To overcome this obstacle in nonlinear RLC circuits the paradigm of *power shaping* was introduced in (Ortega *et al.*, 2003)—where, as suggested by its name, stabilization is achieved shaping the power instead of the energy function. The starting point for the method is a description of the circuit using Brayton–Moser equations (Brayton and Moser, 1964)

$$\mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} = \nabla P + \mathbf{G}(\mathbf{x})\mathbf{u}, \quad (5)$$

where $\mathbf{Q} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ is a *full rank* matrix containing the generalized inductance and capacitance matrices and $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is the circuits mixed potential which has units of *power*, see (Ortega *et al.*, 2003; Jeltsema, 2005) for further details. We make the observation that if $\mathbf{Q} + \mathbf{Q}^\top \leq 0$ then the system satisfies the *power balance inequality*

$$P(\mathbf{x}(t)) - P(\mathbf{x}(0)) \leq \int_0^t \mathbf{u}^\top(\tau) \tilde{\mathbf{y}}(\tau) d\tau$$

with $\tilde{\mathbf{y}} = \tilde{\mathbf{h}}(\mathbf{x}, \mathbf{u})$ and

$$\tilde{\mathbf{h}}(\mathbf{x}, \mathbf{u}) := -\mathbf{G}^\top(\mathbf{x})\mathbf{Q}^{-1}(\mathbf{x})[\nabla P + \mathbf{G}(\mathbf{x})\mathbf{u}]. \quad (6)$$

This property follows immediately pre-multiplying (5) by $\dot{\mathbf{x}}^\top$ and then integrating. The mixed potential function is shaped with the control $\mathbf{u} = \hat{\mathbf{u}}$ where

$$\mathbf{G}\hat{\mathbf{u}} = \nabla P_a \quad (7)$$

for some $P_a : \mathbb{R}^n \rightarrow \mathbb{R}$. This yields the closed-loop system $\mathbf{Q}\dot{\mathbf{x}} = \nabla P_d$, with total power function

$$P_d(\mathbf{x}) := P(\mathbf{x}) + P_a(\mathbf{x}),$$

and the equilibrium will be stable if $\mathbf{x}^* = \arg \min P_d(\mathbf{x})$.

Two key observations are, first, that the resulting controller is *power-balancing*, in the sense that the power function assigned to the closed-loop system is the difference between the total power of the system and the power supplied by the controller. Indeed, from (6) and (7) we have that

$$\dot{P}_a = -\hat{\mathbf{u}}^\top(\mathbf{x}) \tilde{\mathbf{h}}(\mathbf{x}, \hat{\mathbf{u}}(\mathbf{x})) \quad (8)$$

which, upon integration, establishes the claimed property. Second, in contrast with energy-balancing control, power-balancing is applicable to systems with pervasive dissipation. Indeed, in contrast with (4), the right hand side of (8) is *always* zero at the equilibrium, therefore, this equation may be solvable even if $\hat{\mathbf{u}}^\top(\mathbf{x}^*) \mathbf{h}(\mathbf{x}^*) \neq 0$.

As indicated above, instrumental for the application of power-shaping is the description of the system in the form (5). To make the procedure applicable to nonlinear systems described by (1) we apply in the paper Poincaré's Lemma to derive necessary and sufficient conditions to achieve this transformation. We prove in this way that the power-shaping problem boils down to the solution of two linear homogeneous PDEs.

We illustrate the methodology using the textbook example of the tunnel diode (Khalil, 1996). In contrast with the existing techniques for this problem, e.g., approximating the nonlinearities using piecewise-affine functions and convex optimization techniques (Rodriguez

and Boyd, 2005), or designs based on linear approximations (Khalil, 1996), we show that power-shaping yields a simple (partial static state feedback) linear controller that (robustly) globally asymptotically stabilizes the circuit.

2. POWER-SHAPING CONTROL

The main contribution of this paper is contained in the following.

Proposition 1. Consider the general nonlinear system (1). Assume

A.1 There exist a matrix $\mathbf{Q} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $|\mathbf{Q}| \neq 0$, that

i) solves the partial differential equation

$$\nabla(\mathbf{Q}\mathbf{f}) = [\nabla(\mathbf{Q}\mathbf{f})]^\top, \quad (9)$$

ii) and verifies $\mathbf{Q} + \mathbf{Q}^\top \leq 0$.

A.2 There exist a scalar function $P_a : \mathbb{R}^n \rightarrow \mathbb{R}$ verifying

iii)

$$\mathbf{g}^\perp \mathbf{Q}^{-1} \nabla P_a = 0,$$

where $\mathbf{g}^\perp(\mathbf{x})$ is a full-rank left annihilator of \mathbf{g} ,³ and

iv) $\mathbf{x}^* = \arg \min P_d(\mathbf{x})$, where

$$P_d(\mathbf{x}) := \int [\mathbf{Q}(\mathbf{x})\mathbf{f}(\mathbf{x})]^\top d\mathbf{x} + P_a(\mathbf{x}) \quad (10)$$

Under these conditions, the control law

$$\mathbf{u} = (\mathbf{g}^\top \mathbf{Q}^\top \mathbf{Q} \mathbf{g})^{-1} \mathbf{g}^\top \mathbf{Q}^\top \nabla P_a \quad (11)$$

ensures \mathbf{x}^* is a (locally) stable equilibrium with Lyapunov function P_d . Assume, in addition,

A.3 \mathbf{x}^* is an isolated minimum of P_d and the largest invariant set contained in the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \nabla^\top P_d (\mathbf{Q}^{-1} + \mathbf{Q}^{-\top}) \nabla P_d = 0\}$$

equals $\{\mathbf{x}^*\}$.

Then, the equilibrium is asymptotically stable and an estimate of its domain of attraction is given by the largest bounded level set $\{\mathbf{x} \in \mathbb{R}^n \mid P_d(\mathbf{x}) \leq c\}$.

Proof. The first part of the proof consist of showing that, under Assumption A.1, system (1) can be written in the form (5). To this end, invoking Poincaré's lemma we have that (9) is equivalent to the existence of $P : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathbf{Q}\mathbf{f} = \nabla P. \quad (12)$$

Substituting (1) in the above equation and taking into account the full-rank property of \mathbf{Q} in A.1, we get (5) with $\mathbf{G} := \mathbf{Q}\mathbf{g}$.

To prove the stability claim, we proceed as follows. Define $\mathbf{G}^\perp(\mathbf{x}) := \mathbf{g}^\perp(\mathbf{x})\mathbf{Q}^{-1}(\mathbf{x})$, that is a full-rank left annihilator of \mathbf{G} , and the full-rank matrix

$$\begin{bmatrix} \mathbf{G}^\perp \\ \mathbf{G}^\top \end{bmatrix}. \quad (13)$$

³ That is, $\mathbf{g}^\perp \mathbf{g} = 0$, and $\text{rank}(\mathbf{g}^\perp) = n - m$

Left-multiplying equation (5) by (13) yields

$$\begin{bmatrix} \mathbf{G}^\perp \\ \mathbf{G}^\top \end{bmatrix} \mathbf{Q}\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{G}^\perp \nabla P \\ \mathbf{G}^\top \nabla P + \mathbf{G}^\top \mathbf{G}\mathbf{u} \end{bmatrix}. \quad (14)$$

Noticing from (10) and (12) that $P = P_d - P_a$, equation (14) becomes

$$\begin{bmatrix} \mathbf{G}^\perp \\ \mathbf{G}^\top \end{bmatrix} \mathbf{Q}\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{G}^\perp (\nabla P_d - \nabla P_a) \\ \mathbf{G}^\top (\nabla P_d - \nabla P_a) + \mathbf{G}^\top \mathbf{G}\mathbf{u} \end{bmatrix}$$

Now, substituting the control action (11) and iii) of A.2., we finally get the closed-loop dynamics

$$\mathbf{Q}\dot{\mathbf{x}} = \nabla P_d.$$

Taking the time derivative of P_d along the closed-loop dynamics we have

$$\dot{P}_d = \frac{1}{2} \dot{\mathbf{x}}^\top (\mathbf{Q} + \mathbf{Q}^\top) \dot{\mathbf{x}}.$$

Because of ii) of Assumption A.1, $\dot{P}_d \leq 0$ and P_d qualifies as a Lyapunov function. From the closed-loop equation we have $\dot{\mathbf{x}} = \mathbf{Q}^{-1}(\mathbf{x}) \nabla P_d$, hence \dot{P}_d can be rewritten as

$$\dot{P}_d = \frac{1}{2} \nabla P_d \mathbf{Q}^{-\top} (\mathbf{Q} + \mathbf{Q}^\top) \mathbf{Q}^{-1} \nabla P_d$$

Asymptotic stability follows immediately, with Assumption A.3, invoking La Salle's invariance principle. This completes the proof. \triangleleft

Assumption A.1 of Proposition 1 involves the solution of the PDE (9) subject to the sign constraint ii)—which may be difficult to satisfy. In (Ortega *et al.*, 2003) we have proposed a procedure to, starting from a pair $\{\mathbf{Q}, P\}$ describing the dynamics (5), explicitly generate alternative pairs $\{\tilde{\mathbf{Q}}, \tilde{P}\}$ that also describe the dynamics. That is, that satisfy

$$\tilde{\mathbf{Q}}\dot{\mathbf{x}} = \nabla \tilde{P} + \tilde{\mathbf{G}}\mathbf{u}, \quad (15)$$

where $\tilde{\mathbf{G}} = \tilde{\mathbf{Q}}\mathbf{g}$. For ease of reference in the sequel, we repeat here this result adapting the notation to the present context.

Proposition 2. (Ortega *et al.*, 2003) Let \mathbf{Q} be an invertible matrix solution of (9) and define the full-rank matrix

$$\tilde{\mathbf{Q}}(\mathbf{x}) := \left[\frac{1}{2} \nabla (\mathbf{Q}\mathbf{f}) \mathbf{M} + \frac{1}{2} \nabla^\top (\mathbf{M}\mathbf{Q}\mathbf{f}) + \lambda \mathbf{I} \right] \mathbf{Q}, \quad (16)$$

where $\lambda \in \mathbb{R}$ and $\mathbf{M} = \mathbf{M}^\top : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, are arbitrary. Then, system (5) is equivalently described by (15), where

$$\tilde{P} := \lambda \int (\mathbf{Q}\mathbf{f})^\top d\mathbf{x} + \frac{1}{2} \mathbf{f}^\top \mathbf{Q}^\top \mathbf{M} \mathbf{Q} \mathbf{f} \quad (17)$$

Remark 1. Clearly, the power-shaping stage of the procedure—after transformation of the system (1) into the form (5)—is the same as the one proposed in (Ortega *et al.*, 2002) for energy-shaping using interconnection and damping assignment PBC. Additional remarks on the relation between these techniques may be found

in (Jeltsema, 2005; Blankenstein, 2005). It is particularly interesting to note that, viewing power-shaping as control by interconnection with port variables (u, \tilde{y}) , instead of the standard (u, y) (van der Schaft, 2000), yields precisely the interconnection structure proposed in (Maschke *et al.*, 2000)—compare equation (24) in (Blankenstein, 2005) with equation (4.2) in (Maschke *et al.*, 2000).

Remark 2. For port-controlled Hamiltonian (PCH) systems⁴

$$\begin{aligned}\dot{x} &= [\mathbf{J}(x) - \mathbf{R}(x)]\nabla H + \mathbf{g}(x)u \\ y &= \mathbf{g}^\top(x)\nabla H\end{aligned}$$

with *full-rank* matrix $\mathbf{J} - \mathbf{R}$, Assumption A.1 is clearly satisfied with $\mathbf{Q} = (\mathbf{J} - \mathbf{R})^{-1}$. This case has been studied in (Jeltsema *et al.*, 2004), where it is shown that PCH systems satisfy

$$H(x(t)) - H(x(0)) \leq \int_0^t \tilde{y}^\top(\tau)u(\tau)d\tau$$

with the output (6) and an energy-balancing perspective is given to interconnection and damping assignment PBC.

3. THE TUNNEL DIODE

In this section we illustrate the power shaping methodology of Proposition 1 with the benchmark example of the tunnel diode circuit. We show that this technique yields a simple linear controller that ensures robust global asymptotic stability of the desired equilibrium.

3.1 Model Description

Consider the nonlinear circuit of Figure 1 which represents the approximate behavior of a tunnel diode (Khalil, 1996). The dynamics of the circuit is given by

$$\begin{aligned}\dot{x}_1 &= -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{u}{L} \\ \dot{x}_2 &= \frac{1}{C}x_1 - \frac{1}{C}h(x_2)\end{aligned}\quad (18)$$

where x_1 is the current through the inductor L and x_2 the voltage across the capacitor C . The function $h : \mathbb{R} \rightarrow \mathbb{R}$ represents the characteristic curve of the tunnel diode depicted in Figure 2. The assignable equilibrium points of the circuit are determined by $x_1^* = h(x_2^*)$, with the corresponding constant control $u^* = Rh(x_2^*) + x_2^*$. It is easy to see that, for all (non-zero) equilibrium states, the steady-state power extracted from the controller $(u^*x_1^*)$ is nonzero. Consequently, it is not possible to stabilize the circuit via *energy-balancing*. To state our main result we assume the following:

$$\text{A.4 } \min_{x_2} h'(x_2) > -\frac{RC}{L}.$$

⁴ We refer the reader to (van der Schaft, 2000) for a complete treatment on PCH systems.

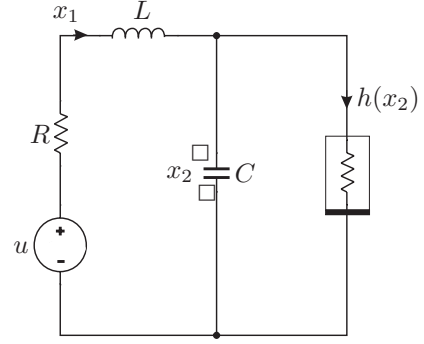


Fig. 1. Tunnel diode circuit.

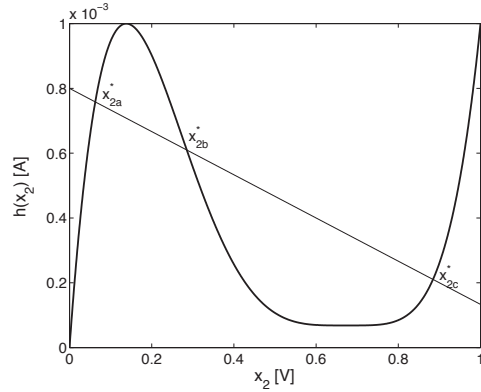


Fig. 2. Tunnel diode characteristic $h(x_2)$ and equilibrium points.

As indicated in Remark 3, this assumption is made for simplicity and can be easily replaced by the knowledge of a lower bound on h' .⁵

3.2 Control Design

Proposition 3. Consider the dynamic equations of the tunnel diode circuit (18), which verifies Assumption A.4. The power-shaping procedure of Proposition 1 yields a linear (partial) state feedback control

$$u = -k(x_2 - x_2^*) + u^*. \quad (19)$$

If the tuning parameter $k > 0$ satisfies

$$k > -[1 + Rh'(x_2^*)], \quad (20)$$

x^* is a *globally asymptotically stable* equilibrium of the closed loop with Lyapunov function

$$\begin{aligned}P_d(x) &= \frac{R}{L} \int_0^{x_2} h(\tau)d\tau + \frac{1}{2C}(x_1 - h(x_2))^2 + \\ &+ \frac{k}{2L}(x_2 - x_2^*)^2 + \frac{1}{2L}(x_2 - u^*)^2.\end{aligned}\quad (21)$$

⁵ We notice that Assumption A.4 coincides with the constraint given in (Moser, 1960) to exclude the appearance of limit cycles in this kind of circuits.

Proof. We look for a matrix $\mathbf{Q}(x)$ such that Assumption A.1 of Proposition 1 is satisfied. Let us propose the simplest form

$$\mathbf{Q}(x) = \begin{bmatrix} 0 & 1 \\ -1 & q(x) \end{bmatrix}$$

where $q : \mathbb{R}^2 \rightarrow \mathbb{R}_{\leq 0}$ is a function to be defined. Computing $\mathbf{Q}\mathbf{f}$, and making q function only of x_2 , we see that the integrability condition (9) reduces to

$$-\frac{1}{C}h' = \frac{R}{L} + \frac{q}{C}.$$

Hence, a suitable matrix is given by

$$\mathbf{Q}(x_2) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{RC}{L} - h'(x_2) \end{bmatrix}, \quad (22)$$

which is invertible for all x_2 and, under Assumption A.4, verifies $\mathbf{Q} + \mathbf{Q}^\top \leq 0$.

Condition iii) of Proposition 1 becomes $\frac{\partial P_a}{\partial x_1} = 0$, indicating that P_a cannot be a function of x_1 . Hence, we fix

$$P_a = \Psi(x_2),$$

where $\Psi(\cdot)$ is an arbitrary differentiable function that must be chosen so that $P_d = P + P_a$ has minimum at x^* . Computing P from (10) we get

$$P_d = \frac{R}{L} \int_0^{x_2} h(\tau) d\tau + \frac{1}{2C}(x_1 - h(x_2))^2 + \frac{1}{2L}x_2^2 + \Psi(x_2),$$

that should satisfy

$$\nabla P_d|_{x=x^*} = \begin{bmatrix} \frac{R}{L}h(x_2^*) + \frac{0}{L} + \Psi'(x_2^*) \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\nabla^2 P_d|_{x=x^*} = \begin{bmatrix} \frac{1}{C} & -\frac{h'(x_2^*)}{C} \\ -\frac{h'(x_2^*)}{C} & \frac{1}{L} + \frac{R}{L}h'(x_2^*) + \frac{(h'(x_2^*))^2}{C} + \Psi''(x_2^*) \end{bmatrix} > 0$$

An obvious selection is then to “complete the squares”

$$\Psi(x_2) = \frac{k}{2L}(x_2 - x_2^*)^2 - \frac{x_2 u^*}{L},$$

for which the conditions above are satisfied provided (20) holds.

Thus, the resulting Lyapunov function P_d is given by (21), which has a unique global minimum at x^* . From (11) we obtain the simple linear state feedback (19). This completes the proof. \triangleleft

To illustrate the general power-shaping procedure we have decided to start from a description of the circuit in the form (1) and explicitly solve the PDE (9). This step can be avoided writing the circuit in Brayton–Moser form (5) and invoking Proposition 2. In this case we have

$$\mathbf{Q} = \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P = - \int_0^{x_2} h(\tau) d\tau + \frac{R}{2}x_1^2 + x_1x_2.$$

We note that the mixed potential P has, indeed, units of power and that $\mathbf{Q} + \mathbf{Q}^\top \not\leq 0$, hence a new pair $\{\tilde{\mathbf{Q}}, \tilde{P}\}$ must be generated. Selecting

$$\mathbf{M} = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{bmatrix}, \quad \lambda = -\frac{R}{L},$$

and invoking Proposition 2, the dynamic equations of the tunnel diode (18) can be equivalently written as (15), with

$$\tilde{\mathbf{Q}}(x_2) = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{RC}{L} - h'(x_2) \end{bmatrix},$$

which correspond to the matrix (22) obtained solving the PDE (9), the new mixed potential

$$\tilde{P}(x) = \frac{R}{L} \int_0^{x_2} h(\tau) d\tau + \frac{1}{2C}(x_1 - h(x_2))^2 + \frac{1}{2L}x_2^2,$$

and $\tilde{\mathbf{G}} = \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix}$. Power shaping is completed along the lines of Proposition 1, but now considering the system in the form (15)

Remark 3. Assumption A.4 is satisfied if the resistance R is large enough. If it is not the case, a possible solution is to apply a preliminary feedback, $-R_a x_1$, with

$$R_a > - \left(\frac{L}{C} \min_{x_2} h'(x_2) + R \right).$$

Remark 4. Knowledge of the system parameters in the power shaping controller (19) is required only for the computation of u^* . Doing a perturbation analysis it can be shown that an error on this constant induces a steady-state error, that can be reduced increasing the gain k —justifying the claimed “robust” qualifier. On the other hand, for all practical purposes, the term u^* can be replaced by an integral action on the voltage error. Current research is under way to study the stability properties of this PI control.

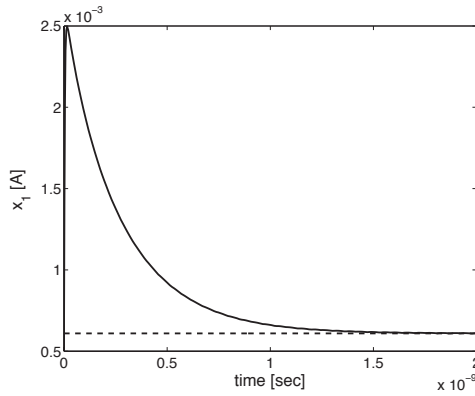
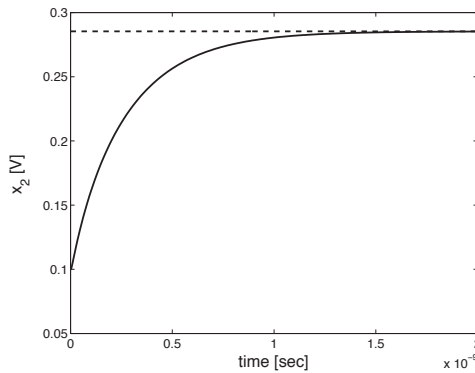
3.3 Simulation Results

The element values of the circuit, taken from (Rodriguez and Boyd, 2005; Khalil, 1996), are $R = 1.5 \text{ k}\Omega$, $L = 5 \text{ nH}$, $C = 2 \text{ pF}$, $u^* = 1.2$ with the currents measured in mA, voltage in Volts and time in nanoseconds. The equation for the characteristic curve of the nonlinear resistor is

$$h(x_2) = 17.76x_2 - 103.79x_2^2 + 229.62x_2^3 - 226.31x_2^4 + 83.72x_2^5$$

It can be proved that Assumption A.4 is satisfied. In open loop the circuit has two stable equilibrium points, corresponding to x_{2a}^* and x_{2c}^* of Fig. 2.

The equilibrium to be stabilized is $x_{2b}^* = 0.2853 \text{ V}$, also indicated in Fig. 2. According to Proposition 3, the gain k of the controller should satisfy $k > 4.46$. The results of simulation are depicted in Figures 3 and 4 with the dashed line representing the desired equilibrium.

Fig. 3. Inductor current x_1 Fig. 4. Capacitor voltage x_2

The initial conditions were set as $x_1(0) = 0.0005$, $x_2(0) = 0.1$ and the gain $k = 5$.

If the equilibrium to be stabilized is x_{2a}^* (or x_{2c}^*), then we have $h'(x_{2a}^*) > 0$ (resp. $h'(x_{2c}^*) > 0$) and the gain condition (20) is satisfied for any $k > 0$ (even some negative values of the gain k do not destabilize these equilibrium points)—this is, of course, consistent with the fact that x_{2b}^* is an (open-loop) unstable equilibrium, while the other two are stable.

4. CONCLUDING REMARKS

In this paper we have extended the *power-shaping* methodology, proposed in (Ortega *et al.*, 2003) for RLC circuits, to general nonlinear systems. We have illustrated this technique with the dynamic model of the tunnel diode. The resulting control law is a simple linear (partial) state feedback controller that ensures (robust) global asymptotic stability of the desired equilibrium point. The simplicity of this controller, which results from the effective exploitation of the physical structure of the system, should be contrasted with the daunting complexity of the “solution” proposed in (Rodriguez and Boyd, 2005). This example, and many other that have been reported in the literature where PBC yields simple sensible solutions, see e.g. (Ortega *et al.*, 1998; Ortega and Garcia-Canseco, 2004) and the references therein, casts serious doubts on the pertinence of piece-wise approximation of nonlinearities to control physical systems.

Among the issues that remain open and are currently being explored are the solvability of the PDE (9) for different kind of systems and other applications of power-shaping, for instance, to mechanical and electro-mechanical systems.

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